Thayer Prize Exam in Mathematics

for First year students

1 - 4 PM Saturday May 9, 2015

Kemeny 105

PRINT NAME: ______________________________

Acknowledgment: Some of the problems are inspired by problems in recent math competitions in the US, Russia, and by problems from other sources.

Honor Code: you are not allowed to give or receive any help on this exam. Using calculators or computers is not allowed.

Grader's use only

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Total: ________ /100

(1) What is the probability that if 5 points are placed randomly on a sphere, some 4 of them will lie in a closed hemisphere?

(2) If n is a positive integer, prove there are 3 distinct positive integers a, b, c with

$$\frac{4}{4n-1} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

(3) Consider a right triangle T with sides of lengths a, b and hypotenuse of length c. Let d be the diameter of the circle inscribed in T. Find the ratio $(a+b)/(c+d)$.

(4) The expression

$$x = \sqrt[3]{\frac{-15}{2} + \frac{23}{6}\sqrt{\frac{11}{3}}} + \sqrt[3]{\frac{-15}{2} - \frac{23}{6}\sqrt{\frac{11}{3}}}$$

is equal to an integer. What is x?

(5) Show that $|\cos(2x) + 3| \geq 4|\cos x|$.

(6) Find all natural numbers n such that the sum of digits of 5^n equals 2^n.

(7) You are given a regular $2n$-gon. Prove that you can put arrows on the sides and diagonals of the $2n$-gon so that the sum of the resulting vectors is 0.

(8) Consider a convex n-gon with equal angles and with consecutive sides $a_1, a_2, \ldots, a_n$ satisfying

$$a_1 \geq a_2 \geq \cdots \geq a_n$$

Show that

$$a_1 = a_2 = \cdots = a_n$$

(9) Each square of a 4×6 chessboard is colored either black or white. The four distinct corner squares of every rectangle (formed by the horizontal and vertical lines of the board) are not of the same color. Prove that there are 12 white and 12 black squares.

(10) Evaluate the integral $\int_{-\infty}^{\infty} x^{2n}e^{-\frac{1}{2}x^2}\,dx$, where n is a positive integer.