Thayer Prize Exam in Mathematics

for Dartmouth First-year students

Saturday May 15, 2021

PRINT NAME: ______________________________

Acknowledgment: Some of the problems are inspired by problems in recent math competitions in the US and in Russia, and by problems from other sources.

Honor Code: You are not allowed to give or receive any help on this exam. Using calculators or computers is not allowed. You have 3 hours to work on the exam and you can choose any 3 sequential hours to do this.

Exam Submission: When you stop working on the exam after 3 hours, you should scan your exam with handwritten solutions into one PDF file and email it back to

Vladimir.Chernov@dartmouth.edu

Your Exam should be received by the end of the day (Eastern Daylight Time) on which you took the exam.

Grader's use only

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Total: _______ /90

(1) On an island there are 17 knights who are obliged to tell only the truth and 18 liars who can only make false statements. All the island's inhabitants were asked to write an essay on the topic of their choice but with an additional requirement that the essay should end either with the sentence "everything written in this essay is true" or with the sentence "everything written in the essay is false". How many essays were submitted by the people on the island, and how did the submitted essays end?

(2) Does there exist a positive integer n such that the fractional part of the number $(2+\sqrt{2})^n$ (that is, the difference of the number and its integer part) exceeds 0.9999999?

(3) What remainders can be obtained when you divide the 100th power of an integer by 125?

(4) Let f be a continuous real-valued function defined on the real line satisfying $f(2x) = f(x)$ for every x. Show that f is constant.

(5) Compute the following limit, if it exists, or explain why it does not exist:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{2i-1}{n}\right)^{1/3}.$$

(6) Let C be a smooth curve in the xy-plane parametrized by

$$(x, y) = (f(t), g(t)), \quad a \leq t \leq b,$$

where $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Show there is a point c in the interval (a, b) such that the tangent line to the curve through the point $(f(c), g(c))$ has slope equal to the slope of the (secant) line through the points $(f(a), g(a))$ and $(f(b), g(b))$. You may assume that none of these lines are vertical.

(7) Consider a collection of $2n$ points in the plane, no three of which are collinear; n of the points are red, while the other n points are green. Show that there is a one-to-one onto function f from red points to green points such that the line segments joining each red point R to its corresponding green point $f(R)$ do not intersect one another.

(8) Let R be a polygon in the plane all of whose vertices are at points whose coordinates are even integers. Show that the area of R is an even integer.

(9) Show that for any integer n, the integers $21n + 14$ and $14n + 3$ have no common prime divisor.