Introductory Explorations into One-Dimensional Poisson-Nernst-Planck (PNP) Systems for Ion Diffusion

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BIOLOGICAL MOTIVATION

The Poisson-Nernst-Planck system is often used to study the flow of several kinds of ions through membrane channels using a set of basic electro-diffusion equations. Membrane channels are part of an essential group of biological membranes that allows the passive transport of ions to and from cells. These membranes, known as the glycocalyx, are highly conserved throughout all organisms and are known to help regulate many interactions between and within cells.

Furthermore, recent developments by Bartozzi et al at Stanford have revealed directions towards promising new immunotherapies through glycocalyx editing. That is, if one is able to control the flow of ions through the glycocalyx, one may be able to boost the immune response towards cancer. Thus, this poster is just the beginning of a project that seeks to understand the PNP system with increasing complexity in order to determine if there exists such opportunities to develop such a controlled system.

FROM 3D TO 1D

The one-dimensional Poisson-Nernst-Planck model is a based on the assumption that the Debye length is much larger than the characteristic radius of the channel—i.e. membrane channels are highly selective, thus allowing the system to be treated as a singularly perturbed set.

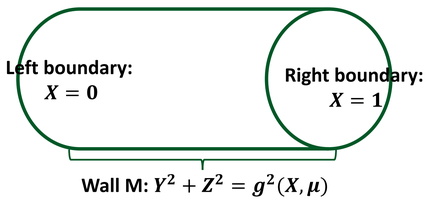

Figure 1: Schematic illustration of our ion channel modeled by $\Omega_\mu = \{(x, y, z): 0 < x < 1, y^2 + z^2 < g^2(x, \mu)\}$, where g is a smooth function satisfying $g(x, 0) = 0$ and $g_0(x) = \frac{\partial g}{\partial \mu}(x, 0) > 0$ for $x \in [0,1]$. The boundary consists of three portions, where the left and right boundaries are viewed as the ends and M is considered the wall of the channel.

3-D	1-D
$\Delta\Phi = -\lambda\left(\sum_{s=1}^{n} \alpha_s c_s + Q\right),$	$\frac{\epsilon^2}{h(x)}\frac{d}{dx}\left(h(x)\frac{d}{dx}\Phi\right) = -\sum_{s=1}^{n} \alpha_s c_s + Q,$
$\frac{\partial c_k}{\partial t} = -\nabla \overline{J_k} = D_k \nabla(\nabla c_k + \alpha_k c_k \nabla\Phi)$	$\frac{dJ_k}{dx} = 0,$
	$h(x)\frac{dc_k}{dx} + \alpha_k c_k h(x)\frac{d\Phi}{dx} = -J_k$
$\Phi\big\vert_{L_\mu} = v_0, \quad c_k\big\vert_{L_\mu} = L_k > 0,$	$\Phi(0) = v_0, c_{k(0)} = L_k;$
$\Phi\big\vert_{R_\mu} = 0, \quad c_k\big\vert_{R_\mu} = R_k > 0,$	$\Phi(1) = 0, c_{k(1)} = R_k$
$\frac{\partial\Phi}{\partial \boldsymbol{n}}\big\vert_{M_\mu} = \frac{\partial c_k}{\partial \boldsymbol{n}}\big\vert_{M_\mu} = 0$	

Table 1: Transformation from the basic 3D PNP system to 1D PNP system with cylindrical symmetry based on the schematic above for $k = 1, \ldots, n$ proposed by Nonner and Eisenberg. Let $\epsilon^2 = \frac{1}{\lambda} \ll 1, h(x) = g_0^2(x)$, and $J_k = \frac{\overline{J_k}}{D_k}$. Permanent charge $Q(x)$ is a piecewise constant function. Boundary conditions are listed below each system respectively.

DIMENSIONAL ANALYSIS

	x	Q	c_k	ϵ	Φ	J_k
T	1	1	0	0	1	1
I	0	1	0	-1/2	2	-4
L	0	0	1	-1/2	-3	1
M	0	0	-2	-1/2	-1	-1

Table 2: Matrix of powers of fundamental units for the six variables. New parameters: x = time, c_k = ion concentration. According to the Buckingham Pi Theorem, there are two dimensionless parameters that can be constructed.

	x	Q	c_k	ϵ	Φ	J_k
T	1	0	0	0	-10/3	16/3
I	0	1	0	0	13/3	-13/3
L	0	0	1	0	-2/3	2/3
M	0	0	0	1	14/3	-2/3

Table 3: RREF Matrix of powers of fundamental units for the six variables revealing two free variables with which the two dimensonless parameters can be constructed.

Letting free variable 1 = 3 and free variable 2 = 0,

$$\pi_1 = \frac{x^{10} c_k^2 \Phi^3}{Q^{13} \epsilon^{14}}$$

Letting free variable 1 = 0 and free variable 2 = 3,

$$\pi_2 = \frac{Q^{13} \epsilon^2 J_k^3}{x^{16} c_k^2}$$

Let k = 1 be Na^{1+}, k = 2 be Ca^{2+}, and $g(x) = e^x$. By introducing $\epsilon\dot{\Phi} = u$, the system can be nondimensionalized into the system below, with properly chosen parameters α, β and γ.

Unscaled Set of ODE	Scaled Set of ODE
$\epsilon\dot{\Phi} = u$	$\dot{p} = \alpha v$
$\epsilon\dot{u} = -(c_1 + 2c_2) - Q(x) - \frac{\epsilon h'(x)}{h(x)} u$	$\dot{v} = -\beta\left(d_1 c_{1_c} + 2d_2 c_{2_c}\right) - \mathrm{Q} - 2x_c v$
$\epsilon\dot{c_1} = -c_1 u - \epsilon J_1 h^{-1}(x)$	$\dot{d_1} = -d_1 v\gamma - J_1 e^{-2\tau x_c} x_c$
$\epsilon\dot{c_2} = -c_2 u - \epsilon J_2 h^{-1}(x)$	$\dot{d_2} = -d_2 v\gamma - J_2 e^{-2\tau x_c} x_c$

Table 4: Comparison between unscaled set of ODE and nondimensionalized set of ODE. Boundary conditions will scale accordingly.

ACKNOWLEDGEMENTS + CITATIONS

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- **Citations**:

1. Liu, Weishi. One-dimensional steady-state Poisson-Nernst-Planck systems for ion channels with multiple ion species. *Journal of Differential Equations 246 (2009).*
2. Liu, Weishi and Wang, Bixiang. Poisson-Nernst-Planck systems for narrow tubular-like membrane channels. *arXiv (2009).*
3. Nonner, Wolfgang and Eisenberg, Bob. Ion permeation and glutamate residues linked by Poisson-Nernst-Planck theory in L-type calcium channels. *Biophysical Journal 75 (Sept 1998).*

QUALITATIVE ANALYSIS

The nondimensionalized system was then analyzed using MatLab.

Varying permanent charge Q and the initial condition V0, four conditions were tested for each equation. Interesting changes in systemic behavior were observed, including multiple internal layers and different equilibria as time approached infinity based on these different conditions.

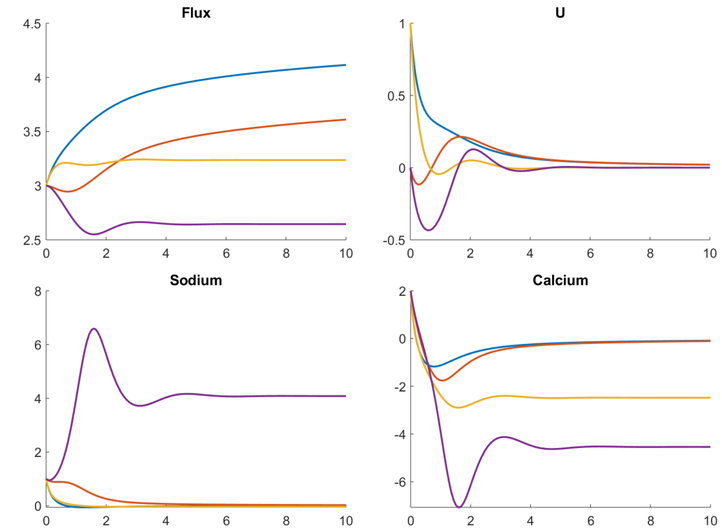

Figure 2: The four above plots illustrate the different behaviors due to different starting conditions based off of the boundary conditions. P, C1, and C2 were all assumed to be non-zero and positive. Blue: Q = 0, U = 1. Red: Q = 0, U = 0. Yellow: Q = 1, U = 1. Purple: Q = 1, U = 0. It is interesting to see that flux (UL) reaches a different equilibrium with every different set of starting conditions while U (UR) consistently converges to 0 regardless of the case tested. Sodium seems similarly unperturbed until only Q is nonzero, at which case it stabilizes above 0. Calcium seems to reflect the opposite trend.

Boundary layer analysis for the cases involving Q = 0 has already been conducted by Liu and Wang in their analysis. Internal layer analysis is beyond the scope of this course, but would be an interesting future direction. Clearly, some variables are more sensitive to change than others. As such, it would be beneficial to conduct some sensitivity analysis in order to roughly predict how the behavior of this system changes upon perturbation.

FUTURE WORK

1D PNP system matches previous data regarding the existence of shock layers as permanent charge increases → Learn more about PDEs in order to conduct shock layer and sensitivity analysis on system → Mathematically explore possibilities regarding controlled systems using PNP models to match active transport data