Modeling the Tibetan Singing Bowl

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Introduction

The Tibetan Singing Bowl has been used for centuries across the world for therapeutic and meditative purposes. It has been proven to improve mood and reduce stress levels. The bowls are typical made of a bronze alloy consisting of copper, tin, zinc, iron, gold, and nickel, and is played by striking or rubbing a wooden or rubber mallet on the outer rim of bowl. This produces a singing-like, meditative tone.

In this project, we attempted to model the vibrations of the Tibetan Singing Bowl to analyze the frequency produced when the bowl is struck with the mallet. The unique, curved shape of the bowl's rim introduces a complication in modeling the bowl (Figure 2 shows one attempt by Golas et. al.) Further, Tibetan Singing bowls are not uniform, with varying alloy compositions, shapes, and wall thicknesses. These features necessitated simplifications in modeling the bowl. We approximate the shape with that of a sphere. We also made the assumption that in our hypothetical spherical model, the vibrations of the air inside the "sphere" had a minimal effect on the frequence produced.

Figure 1: Tibetan Singing Bowl and Mallet

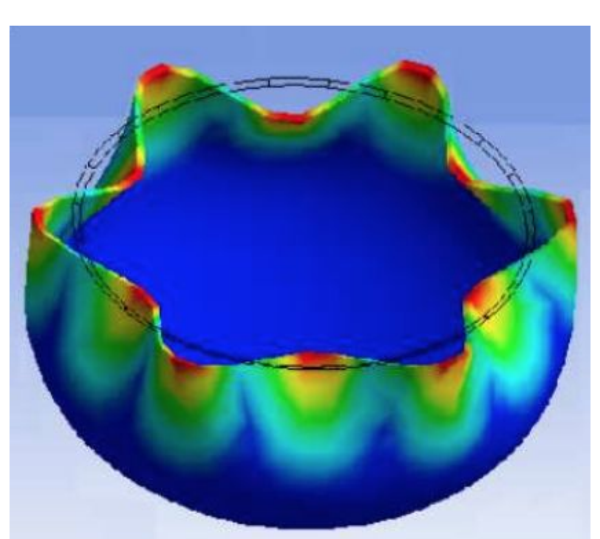

Figure 2: Digital Model of one of many mode shapes for the Bowl (Golas)

Methods

$$\Delta_s u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$u(r,\theta,\phi,t) = f(t)g(\theta,\phi); g = h(\theta)s(\phi) \tag{2}$$

$$\frac{1}{c^2}\frac{d^2 f}{dt^2}\frac{1}{f} = \frac{\Delta_s g(\theta,\phi)}{g} = \lambda \tag{3}$$

$$\frac{d^2 s}{d\phi^2} = -m^2 g \tag{4}$$

$$\frac{1}{sin(\theta)}\frac{d}{d\theta}(sin\theta\frac{dh}{d\theta}) + [\lambda - \frac{m^2}{sin^2\theta}]h = 0, \lambda = -n(n+1) \tag{5}$$

$$\frac{1}{c^2}\frac{1}{f}\frac{d^2 f}{dt^2} = -n(n+1) \tag{6}$$

$$u(r,\theta,\phi,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(cos\theta)e^{im\phi}(A_{mn}cos(q_n t) + B_{mn}sin(q_n t)) \tag{7}$$

Equation	Meaning / Significance
1	Wave Equation for a Sphere
2	Re-writing the Function as Product of Time and Spatial Equations
3	Separation of Time and Spacial Variables in the Wave Equation
4	Equation in Phi
5	Theta Equation: Gives lambda = -n(n+1)
6	Time Variable Set Equal to Lambda
7	General form of the solution. P is the Legendre Family of Polynomials

Since our model is looking at the mechanisms of the Tibetan singing bowl as if it were a sphere, we started with wave equation in spherical polar coordinates. We separated the function "u" into spatial and time variables, and further separated the spatial variables into the product of univariate equations, as shown in Equation 2. Since we modeled the bowl as the boundary of a sphere, the radius is constant which means only the angular components of the spatial variables are of interest. After taking the spherical laplacian of u, we isolated the time variable from the spatial coordinates and then further separated the spatial coordinates in Equations 4 and 5. A Fourier series of these solutions is shown in Equation 7 which sums all linearly independent solutions, giving the general solution.

To estimate the frequencies of the bowl, equations 6 was of particular interest, as frequencies of the time component are the ones we hear. Solving Equation 5 restricts lambda to the form -n(n+1), where n is an integer. Plugging into Equation 6 gives a family of sinusoidal function dependent on n. The frequencies of these functions are given by some constant c times the square root of n(n+1). The constant c depends on the exact composition of the alloy, and, since this is unknown to us, is not able to be calculated to estimate the frequencies we should hear. However, without knowing the constant, we are still able to model the relative spacings between the observed frequencies to test whether or not the bowl is behaving similarly to a sphere. Per the discussion above, the frequencies should be spaced in proportion to the square root of n(n+1). The results section shows frequency estimates in black bars superimposed on the actual frequency distribution recorded with Friture.

Results

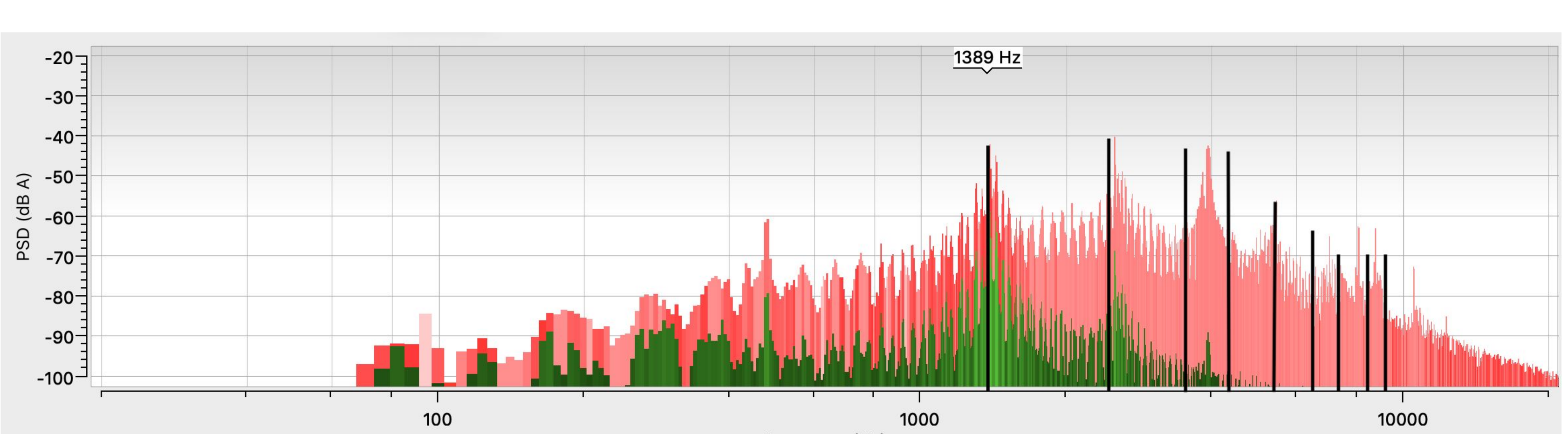

Figure 3: Frequency achieved from striking the bowl analyzed by Friture. The black lines indicate our predictions.

The first peak lines up exactly, as this was used to represent the case where n=1, and thus the rest of the frequencies should be spaced relative to this fundamental frequency. Subsequent frequency estimates were close to the observed frequencies, but slightly off due to the simplifications made in the modelling process. These include the shape of the bowl, the initial and boundary conditions, and the behavior of air inside of the bowl, as mentioned in the introduction. Still, the relative spacing of the estimated frequencies aligned well with observation, with a maximum error of 8.2% for a single spacing observation and an average error of 4.2%. Note, compounding errors create larger absolute errors than the individual spacing errors. Interestingly, the estimated spacings were underestimates for the first four frequencies, but were overestimates thereafter. This suggests a simple scaling of the model would not improve the estimates, and a more complicated model would be necessary to estimate the frequencies more precisely. In all, the bowl behaved approximately as a sphere, but the unique attributes of the singing bowl introduce complexities and thus error into the system as modeled.

References

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3. Goldsby, Tamara L et al. "Effects of Singing Bowl Sound Meditation on Mood, Tension, and Well-being: An Observational Study." *Journal of evidence-based complementary & alternative medicine* vol. 22,3 (2017): 401-406.
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Acknowledgments

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