Modeling Chladni Plates Using Neumann Boundary Conditions

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Abstract

Our research aims at mathematically describing the Chladni Pattern for a plate based on the wave equation in two dimensions. It begins with constructing a model using Neumann boundary condition, and gives the theoretical general solution through the methods of separation of variables. The next step is testing the nodal pattern with real Chladni's plate to derive the empirical results. The equations of nodal patterns depend on the nodal numbers m and n which were related with the frequency of the driving force, which in our case is by striking with a bow. Also in many situations the patterns are a mixture of some basic forms of patterns. Our model succeeded in satisfying all conditions but failed to predict the resonant frequencies and give a single value for the tension constant: c.

Introduction

History

In the early 18 century, German mathematician Ernst Chladni repeated the Robert Hooke's pioneering experiment. He successfully invented a technique to see various vibration modes on a rigid surface. He used sand sprinkled on a plate to show nodal lines, which has attracted attention of many famous figures. It was believed that Napoleon was so delighted with Chladni's work that he financed Chladni's publication to be translated into French.

Chladni's law for vibrating plates is later further studied by several scholars. A female mathematician Sophie Germain won a contest relating to vibration theory. In 1850, Kirchhoff gave more accurate boundary conditions. Later, another female mathematician Mary Waller further modified the hypothesis. Together, we get the model we use today.

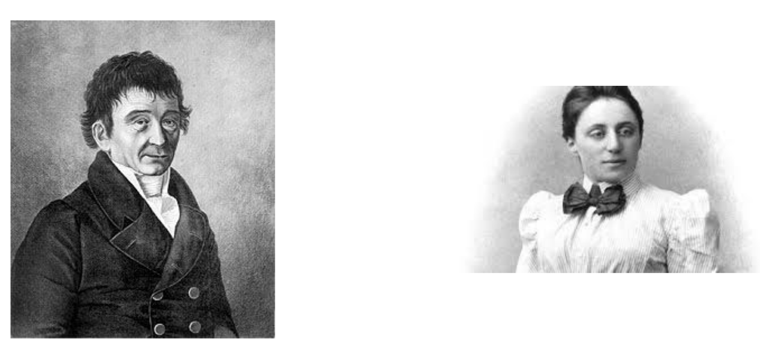


Figure 1&2: Ernst Chladni and Sophie Germain

Introduction to the Device

Modern Chladni plate has several model devices. The most commonly seen one is a plate attached to a mechanical driver in the middle. By changing the frequency and amplitude of the driver, we can observe different patterns of the sand. However, by attaching to a driver at the center, it greatly limits the movements and possible nodes we can get from the Chladni plate.

Another model we see is a plate simply pinned down in the middle. We will strike it with or without a bow. Not only can it produce some "melody", it could also potentially give us a pattern of nodes and antinodes. However, it is harder to produce a sustained frequency that will lead to a pretty pattern.

How we plan to test the Chladni's law

Step 1: Construct theoretical model
Step 2: Solve for theoretical results
Step 3: Confirm with empirical results

Theory

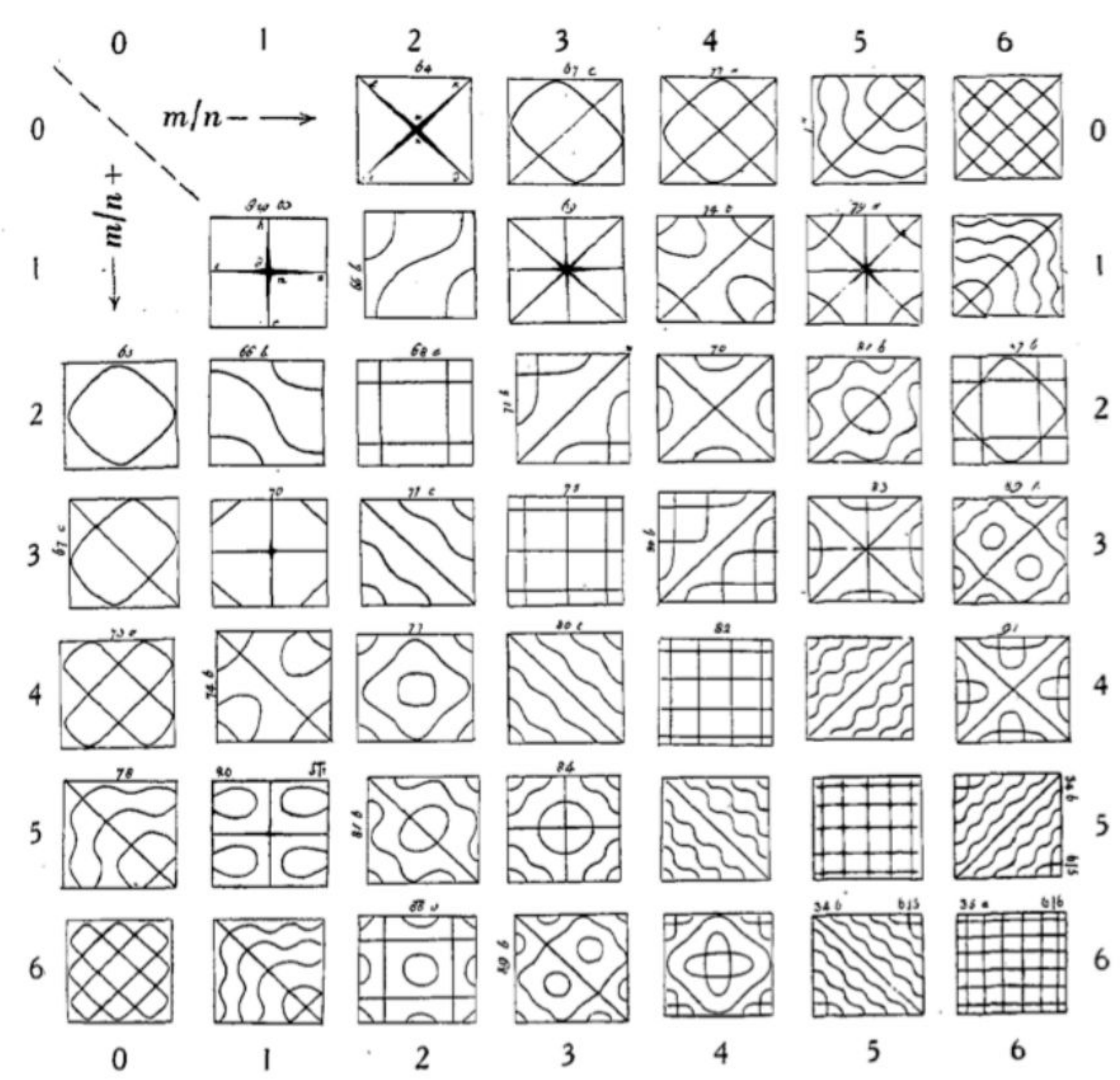
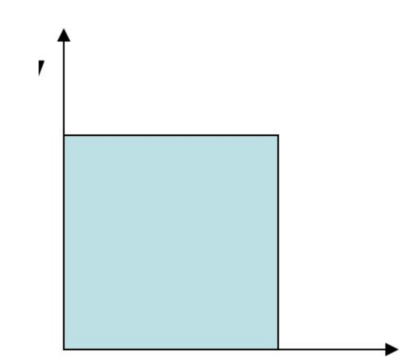


Figure 2: Waller's finding of Cladni's patterns

Theoretical Model

The model was developed with the following conditions:
u: displacement of the plate, x: width, y: height, t: time



Model of the wave equation: $c^2\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Neumann boundary condition: $u_t(0,y,t)=0 \quad u_t(x,0,t)=0$ (must be anti-node at plate edges)

More boundary conditions: $u(12,12,t)=0$ (the center is pinned and thus is stationary forever)

Initial conditions: $u(x,y,0)=0$

Separation of variables: $u(x,y,t)=f(x)g(y)h(t)$

Theoretical Result

After applying to the conditions of our plate: 0<x<24, 0<y<24

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}sin(\tfrac{\pi c}{24}t\sqrt{m^2+n^2}))(cos(\tfrac{n\pi}{24}x))(cos(\tfrac{m\pi}{24}y))$$

$$A_{mn} = \frac{4}{24^2}\int_0^{24}\int_0^{24} f(x,y)cos(\tfrac{n\pi}{24}x)cos(\tfrac{m\pi}{24}y)dxdy$$

Empirical Result

We manage to get three different patterns with our Chladni plate, each achieving a different frequency. Our fundamental frequency (figure 3) is 387 Hz. Two other patterns have frequency 1893 Hz (figure 4) and 5250 Hz (figure 5). Due to the limitation of striking it with a bow, we are unable to produce more different patterns. But various kinds of patterns have been observed by previous scholars (for detail see figure 2).

Our empirical results confirm our theoretical results, although several modification could be made to perfect our model.

1. *As we do not know the exact material of the plate, it is hard to predict c with our observations. Right now we are using an estimation, but more accurate material condition could potentially strengthen the accuracy of our theoretical model.*
2. *Operating our empirical model in a quieter setting and with more accurate devices could help eliminate the "noise" in our model, which would also strengthen the match of our theoretical results and empirical results.*

Figure 3 & 4 & 5: Our finding of Chladni's patterns

Conclusion

From calculation of fundamental frequencies the values for c were found to be different in each case. The values for c were 1478, 3233, and 6878 cm·s for all of them. Given these results, the Neumann boundary conditions do not seem to be the best for for predicting the behavior of a Chladni plate since the c should be constant among the resonant frequencies.

The original boundary conditions that Sophie Germain used in her answer may be the closest we get to modeling a Chladni plate. However, her boundary conditions were not found during the research for the project. In the future, these boundary conditions along with more material science input could prove to aid the model. In the future, modeling this phenomenon on programs with graphing capabilities such as MatLab could help in visualizing the nodes in 2D.

References

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