The Heat Equation of Flat Two- and Three-Dimensional Objects

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INTRODUCTION

The heat equation is a partial differential equation (PDE) that describes the distribution of heat over time. It was developed by Joseph Fourier in 1822 and has become crucial in many applications of physics and mathematics. The general form of the heat equation is given by:

$u_t = k[u_{xx}]$, where k is a positive constant

Using the two- and three-dimensional equivalents of this equation, this project aims to model the distribution of heat on the torus and the solid cube. It also explores potential methods for solving the heat equation of the möbius strip.

METHOD: TORUS

To analyze the heat distribution on the torus, it is necessary to look at its 2D representation as an unfolded rectangle. Therefore, the heat equation in 2D, given by $u_t = k[u_{xx} + u_{yy}]$, is used. From the rectangle, we obtain the Neumann boundary conditions:
$u(0,y,t) = u(L,y,t)$, $u(x,0,t) = u(x,J,t)$ [see figure 2].
Assume solution of the form $u(x,y,t)=f(x)g(y)h(t)$. From the 2D heat equation, and by separation of variables, the equation becomes:

$$k\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right] = \frac{h'(t)}{h(t)} = \lambda_1 + \lambda_2 = \lambda_{n,m}$$

Evaluating each spatial variable individually to find its corresponding eigenvalue and eigenfunctions $f(x)$ and $g(y)$ allows for solving for $h(t)$, the function in time. Once all three functions have been found, $u(x,y,t)$ is written as a summation for all positive integers n and m of the product $f(x)g(y)h(t)$.

RESULTS: TORUS

The general solution found is:

$$u(x,y,t) = \sum_{n,m \in Z^+} a_{n,m} exp[-k\lambda_{n,m}t]cos(\tfrac{2n\pi x}{L})cos(\tfrac{2m\pi y}{J})$$

where,

$$\lambda_{n,m} = -\left[\frac{4n^2\pi^2}{L^2} + \frac{4m^2\pi^2}{J^2}\right]$$

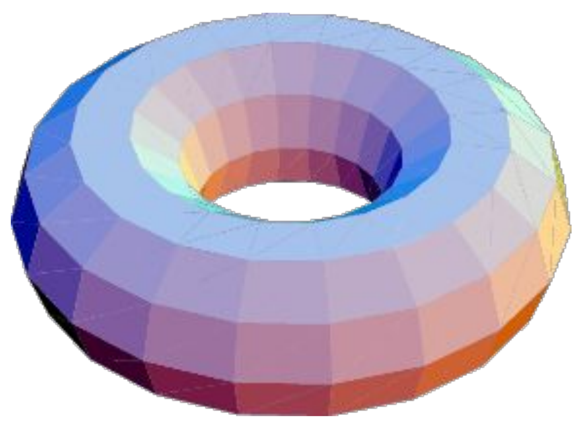

Figure 1: The torus

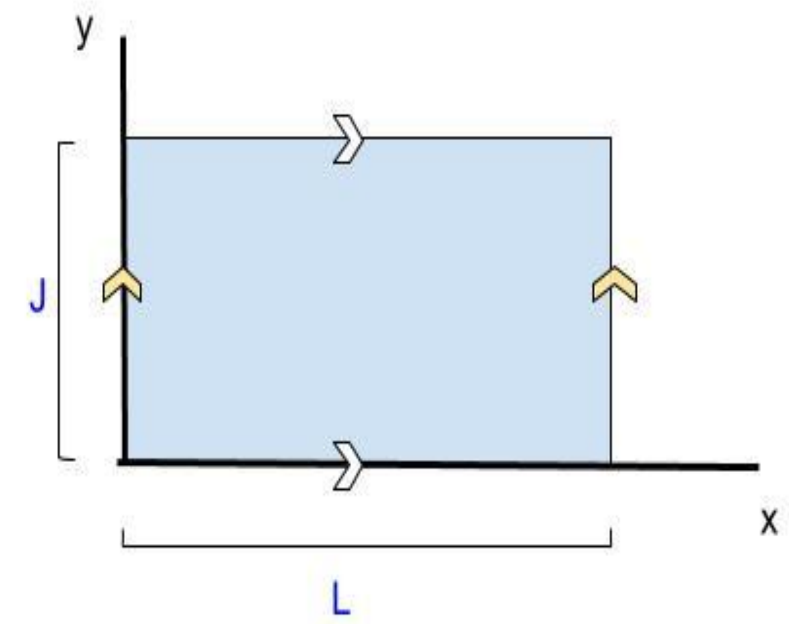

Figure 2: The 2D representation of the torus

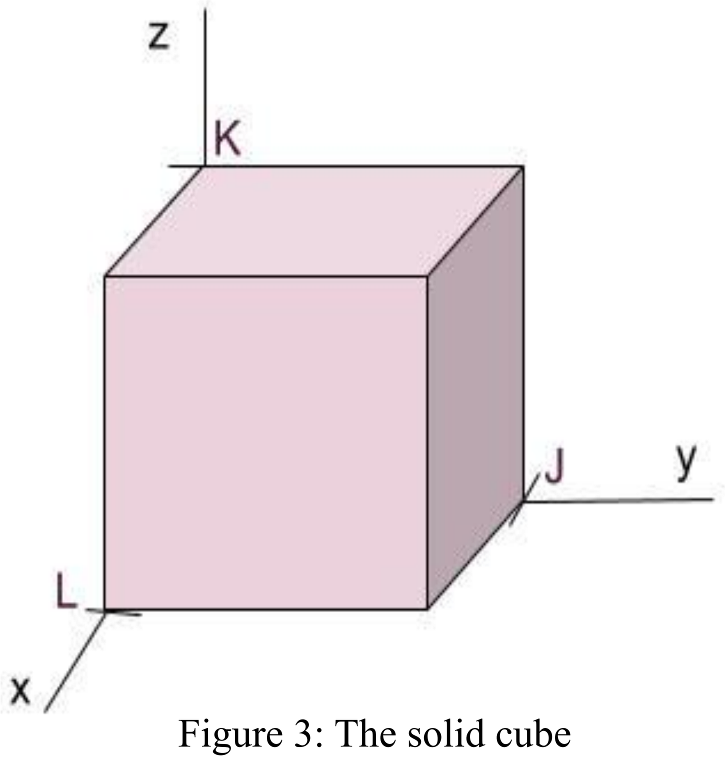

Figure 3: The solid cube

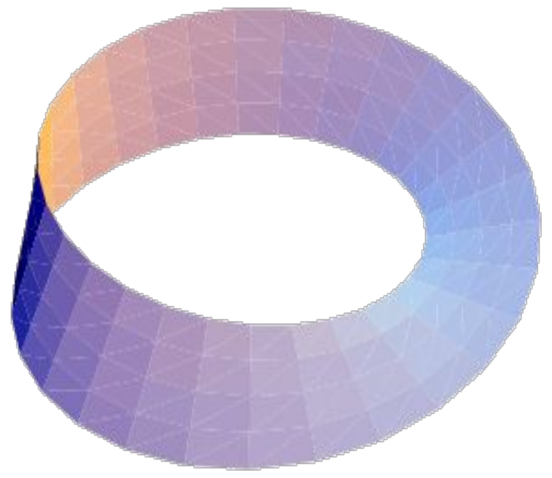

Figure 4: The möbius strip

Figure 5: The 2D representation of the möbius strip

ACKNOWLEDGMENTS AND REFERENCES:

(1) Professor Dorothy Wallace, Department of Mathematics, Dartmouth College
(2) Logan, David. *Applied Mathematics*. Fourth ed., John Wiley & Sons.
(3) Boyce, William E.,DiPrima, Richard. C. *Elementary Differential Equations and Boundary Value Problems.*Tenth ed.,John Wiley & Sons.
(4) https://gauss.math.yale.edu/~mr2245/pdespr2017Data/LecNotes.pdf
(5) https://physics.bgu.ac.il/~dcohen/ARCHIVE/mbs_PRB.pdf

METHOD: CUBE

Given a solid cube of volume V [see figure 3] that is insulated on the boundary dV, it is necessary to use the heat equation in three dimensions, $u_t = k[u_{xx} + u_{yy} + u_{zz}]$, to model its heat distribution. The insulated boundary gives Dirichlet boundary conditions: $f'(0)=f'(L)=0$, $g'(0)=g'(J)=0$, $h'(0)=h'(K)=0$. Assume solution of the form $u(x,y,z,t)=f(x)g(y)h(z)p(t)$. From the 3D heat equation, and by separation of variables, the equation becomes:

$$k\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)}\right] = \frac{p'(t)}{p(t)} = \lambda_{n,m,q}$$

Evaluating each spatial variable individually to find its corresponding eigenvalue and eigenfunctions $f(x)$, $g(y)$, $h(z)$ allows for solving for $p(t)$, the function in time. $u(x,y,z,t)$, the general solution for the heat distribution on the cube is the sum, for all positive integers n,m,q, of the product of $f(x)g(y)h(z)p(t)$.

RESULTS: CUBE

The general solution found is, $u(x,y,z,t)$ =

$$a_0 + \sum_{n,m,q \in Z^+} a_{n,m,q} exp[-k\lambda_{n,m,q}t]sin(\tfrac{2n\pi x}{L})sin(\tfrac{2m\pi y}{J})sin(\tfrac{2q\pi z}{K})$$

where,

$$\lambda_{n,m,q} = -\left[\frac{4n^2\pi^2}{L^2} + \frac{4m^2\pi^2}{J^2} + \frac{4q^2\pi^2}{K^2}\right]$$

MÖBIUS STRIP

Treating the möbius strip as a 2D shape [see figure 5], the boundary conditions that follow are: $u(0,0,t)=u(L,J,t)$, $u(0,J,t)=u(L,0,t)$.
However, neither boundary condition holds a spatial variable constant, and therefore, requires a different method of solving. Forcing Dirichlet boundary conditions $f'(0)=f'(L)=0$ allows for a solution for $f(x)$. Assume

$$g(y) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(y-\lambda)^2}{4kt}} cos(\sqrt{\lambda}\, x) \quad .$$

However, evaluating $\triangle f(x)g(y)=ku_t$ reveals $f(x)g(y)$ is not a solution of the wave equation. Although no general solution was found, the möbius strip is an interesting application of treating a 3D object in its 2D representation.