A Game Theoretic Model of Jazz Improvisation

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Abstract

This paper presents a novel approach to modeling jazz improvisation as a mathematical game. Such a game provides insights into music theory and provides a new method of conceptualizing the practice of musical improvisation in general. Through computational modeling, we uncover the underlying payoffs associated with diverse music strategies including randomness, chord following, and reinforcement learning to name a few.

Introduction and Methods

Background

Much of music theory has been devoted to the understanding musical perception from the lens of mathematics. This study has provided many useful insights that will be utilized in our paper.

First, harmony has been historically measured ranging from consonance to dissonance. This range is a measure of how good two notes sound to our ears, with consonant harmony typically sounding more harmonious and vice versa. Nearly all the information of whether two notes sound harmonious or not comes from the base frequencies of each note. We will use one of the leading theories on analyzing harmony, that involves measuring the ratios of each frequency.

Grace Leslie & Navid Hassanpour (2008) have proposed that simple improvisation can be modeled using a simple game in which each player gets to pick between two possible notes. From this, it is found that a game theoretic model of musical interaction is possible and can be represented by payoff matrices. Our game is a significant expansion upon this idea. We present each player with 4159 options representing every integer frequency found in between the lowest note on the piano and the highest. Every "beat" (which is equivalent to a turn in the game), both players pick a note without knowledge of what the other player is going to play. Then, a payoff is calculated based on these choices. The rules to how these payoffs are calculated are founded upon the latest research in music theory.

For the sake of simplicity, we will analyze jazz improvisation using a very common jazz chord progression: the 12 bar Bb blues: Bb7, Eb7, Bb7, Bb7, Eb7, Eb7, Bb7, Bb7, Cm7, F7, Bb7, F7 (Other chord progressions and other variations of the blues could be used).

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Findings

	R	CF	SF	HP	SC	SRL	CFRL	CSR	TPRL
R	0.06506	-	-	-	-	-	-	-	-
CF	0.24063	0.47265	-	-	-	-	-	-	-
SF	0.18968	0.40900	0.35242	-	-	-	-	-	-
HP	0.07731	0.42821	0.28993	0.08311	-	-	-	-	-
SC	0.16330	0.53316	0.33924	0.56106	0.47824	-	-	-	-
SRL	0.24602	0.49052	0.43848	0.39985	0.44084	0.50365	-	-	-
CFRL	0.24934	0.49433	0.42602	0.45701	0.54871	0.51148	0.51715	-	-
CSR	0.22686	0.46745	0.40906	0.37266	0.42285	0.48545	0.50001	0.45814	-
TPRL	0.21346	0.47801	0.43093	0.32752	0.46160	0.50782	0.50827	0.48574	0.52588

Table: R: Randomness, CF: Chord Following, SF: Scale Following, HP: Harmony Prediction, SC: Stepwise Changes, SRL: Simple Reinforcement Learning, CFRL: Chord-Following Reinforcement Learning, CSR: Chord-Specific Reinforcement Learning, TPRL: Two-Player Reinforcement Learning

Figure 1: Payoff Matrix after 10 Simulated Trials

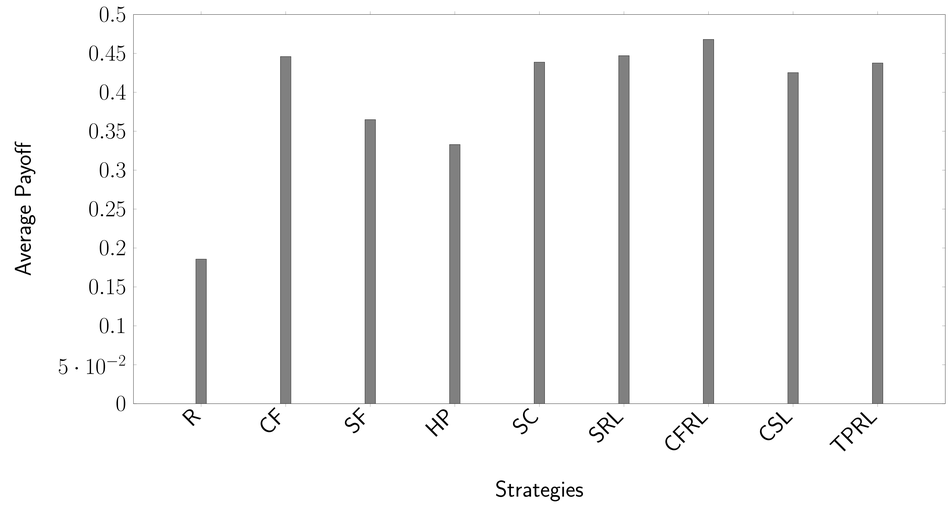

Figure 2: Average Payoff by Strategy

Strategies

- Randomness
- Chord Following
- Scale Following
- Harmony Prediction
- Stepwise Changes
- Simple Reinforcement
- Chord Reinforcement
- Scale Reinforcement
- Two-Player Reinforcement

Payoff

Two scores were used to calculate a payoff for each game: variance and harmony.

1. The **variance** score, denoted by V, utilizes Shannon's Diversity Index and species evenness as a proxy for how the players vary what notes they play.
2. The **harmony** score, denoted by H, uses six notes, one played by each player and four representing the current chord in the chord progression. Each possible pair of these six notes is used in a fraction, which is then simplified. These numerator/denominator sums are added and averaged to get the harmony score, where a smaller harmony score is better.

The payoff is represented by:

$$P = \frac{V - H}{V + H}$$

Results

After ten trials of each strategy vs. strategy pair (45 pairs), the payoff of each individual trial was calculated. The average of these trials was taken and is displayed in Figure 1 (Table). The average of each strategy's average payoff in each pair was then also taken and displayed in Figure 2.

Conclusion

The reinforcement learning algorithms and the stepwise changes strategy, which all depend on previous payoffs, tended to perform similarly well. Chord-based strategies, including chord following and the chord-following reinforcement learning strategies, also did performed very well.

Discussion

Reinforcement learning and stepwise change strategies performing well indicates that musicians in the game should be open to switching based on new feedback. In real life, this would correlate to a musician shifting strategies after hearing a few unharmonic beats of music.
Furthermore, chord-following strategies performing well should indicate that musicians should emphasize playing music that would harmonize well with the current chord. Since the chord-following reinforcement learning strategy had the highest average payoff out of all tested strategies, having different notes to play based on the current chord is a viable strategy for "better" (as defined by our simplified model) music.

Future Research

Quantifying music yields the use of several different models, and the model used in this paper utilizes several simplifying assumptions. While this yields some interesting results, future research may include models accounting for rhythm, melody, and other factors. Furthermore, the score calculations themselves could be adjusted to value how a certain type of music counts through curves based on present-day music.
We can also make a network game, with a population of musicians with different strategies, and switch strategies of musicians that have low average payoffs after their interactions (2-player jazz improvisation games).

Figure 3: QR Code for chosen samples of generated music

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